Tutorial 10: Selected problems of Assignment 9

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Q1) (Supp. Ex. 3)

Show that $S(x) := \sum_{j=1}^{\infty} \frac{\cos 2^j x}{3^j}$ is a continuous function on $\mathbb{R}$.

Determine whether $S(x)$ is differentiable on $\mathbb{R}$

Pf) Note that for all $j \in \mathbb{N}$, for all $x \in \mathbb{R}$

$$\left| \frac{\cos 2^j x}{3^j} \right| \leq \frac{1}{3^j}, \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{3^j} \text{ is finite}$$

$\therefore$ By M-test (Thm 3.10), $\left(S_n(x) := \sum_{j=1}^{n} \frac{\cos 2^j x}{3^j} \right)$ converges uniformly to $S(x)$ on $\mathbb{R}$. Since for all $n \in \mathbb{N}$, S_n is continuous on $\mathbb{R}$, by Continuity Theorem (Thm 3.6)

S is continuous on $\mathbb{R}$

consider $S_n'(x) = \sum_{j=1}^{n} \left(\frac{-\sin 2^j x}{3^j} \cdot 2^j \right) = -\sum_{j=1}^{n} \left(\frac{2}{3}\right)^j \sin 2^j x$

Then as $\sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^j$ is finite, by M-test, S_n' converges uniformly on $\mathbb{R}$

Therefore, by Differentiability Theorem (Thm. 3.8),

$S(x)$ is differentiable on $\mathbb{R}$.

Q2) (Supp. Ex. 4) Let $S_n : [1, +\infty) \longrightarrow \mathbb{R}$ be defined as

$$S_n(x) := \sum_{j=0}^{n} e^{-jx}.$$

(a) Show that S_n converges uniformly on $[1, +\infty)$ and $S(x) := \sum_{j=0}^{\infty} e^{-jx}$ is smooth on $[1, +\infty)$

(b) Show that S_n does not converge uniformly on $[0, +\infty)$

Pf) (a) Note that for all $j \in \mathbb{N}$, $x \in [1, +\infty)$,

$$e^{jx} \geq \frac{(jx)^2}{2!}, \quad \therefore \quad e^{-jx} \leq \frac{2}{j^2x^2} \leq \frac{2}{j^2}$$

Since $1 + \sum_{j=1}^{\infty} \frac{1}{j^2}$ is finite, by M-test,

S_n converges uniformly to S.

Note that for each $x \in [1, +\infty)$,

$$S(x) = \sum_{j=0}^{\infty} e^{-jx} = \frac{1}{1-e^{-x}},$$

$\therefore$ $S(x) = \frac{1}{1-e^{-x}}$ is smooth on $[1, +\infty)$

(b) Note that when $x = 0$, $S_n(0) = n+1$, $\therefore$ $\lim_{n\to\infty} S_n(0)$ diverges

Therefore, S_n does not converge pointwisely, hence uniformly, on $[0, +\infty)$.

Q3) (Supp. Ex. 5) Let $(f_j : E \to \mathbb{R})_j$ be defined

and $S_n := \sum_{j=1}^{n} f_j$ Converges pointwisely to $S(x)$ $\left(= \sum_{j=1}^{\infty} f_j(x)\right)$

(a) Then for any $g : E \to \mathbb{R}$, for all $x \in E$,

$t_n(x) := (g S_n)(x) = \sum_{j=1}^{n} g(x) f_j(x)$ Converges pointwisely to $t(x) := g(x) S(x)$

(b) If (S_n) converges uniformly to S, and g is bounded,

then t_n converges uniformly to t on E

Pf) (a) For any $x \in E$, $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|S_n(x) - S(x)| \leqslant \frac{\varepsilon}{1 + |g(x)|}$$

then $|t_n(x) - t(x)| = |g(x) S_n(x) - g(x) S(x)|$

$$= |g(x)| \, |S_n(x) - S(x)| \leqslant |g(x)| \cdot \frac{\varepsilon}{1+|g(x)|} < \varepsilon$$

$\therefore$ $t_n(x)$ converges to $t(x)$ pointwisely

b) Since g is bounded, there exists $M \in \mathbb{R}$ such that

for all $x \in E$, $|g(x)| \leq M$

$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall x \in E$,

$$|S_n(x) - S(x)| \leqslant \frac{\varepsilon}{1+M}$$

then $|t_n(x) - t(x)| = |g(x)| |S_n(x) - S(x)| \leqslant M \cdot \frac{\varepsilon}{1+M} < \varepsilon$

$\therefore$ t_n converges uniformly to t

Remark: Note that same conclusions hold for any sequences of function $S_n : E \to \mathbb{R}$, not necessarily a sequence of partial sums of functions.

Q4) (Supp. Ex. 8) Let $S_n : \mathbb{R} \to \mathbb{R}$ be defined as

$$S_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$$

Show that S_n does not converges uniformly on $\mathbb{R}$.

Pf) Suppose on the contrary S_n converges uniformly on $\mathbb{R}$.

Then by Cauchy Criterion (Thm. 3.4)

For $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $\forall x \in \mathbb{R}$,

$$|S_{N+1}(x) - S_N(x)| \leq 1$$

$\therefore \left| \frac{x^{N+1}}{(N+1)!} \right| \leq 1$, for all $x \in \mathbb{R}$

which is a contradiction, e.g. choose $x = N+1$

Rmk: Note that S_n converges uniformly on any bounded intervals.

(Thm 3.11 (a))